ECS 332: Tutorial

1) [M2018] Evaluate the following integral: $\int_0^1 \delta\left(e^{2t}-2\right)dt$

Method 1: Use integration by substitution (change of variables):

Let $u = e^{2t} - 2$. Then, $t = \frac{1}{2}\ln(u+2)$ and $\frac{dt}{du} = \frac{1}{2}\frac{1}{u+2}$. Therefore,

$$\int_0^1 \delta(e^{2t}-2)\,dt = \int_{u=e^{2t}-2|_{t=0}}^{u=e^{2t}-2|_{t=1}} \delta(u)\frac{1}{2}\frac{1}{u+2}du = \int_{-1}^{e^2-2} \frac{1}{2}\frac{1}{u+2}\delta(u)\,du$$

Now, $e^2 - 2 \approx 5.3891$. Therefore, $u = 0$ is inside the range of integration.

From the sifting property of the δ-function, the integral becomes

$$\left.\frac{1}{2}\frac{1}{u+2}\right|_{u=0} = \frac{1}{4}.$$

Method 2: When we deal with δ-function, we know that only the value at 0 of its argument matters. So, we will try to see how $e^{2t}-2$ behaves near the value of t that makes it close to 0 which is $t_0 = \frac{1}{2}\ln(2) \approx 0.3466$.

Recall, from calculus, that any nice function can be approximated by a straight line if we consider only small region. In particular, for t near t_0, a function $g(t)$ can be approximated by

$$g(t) \approx g'(t_0)(t-t_0) + g(t_0).$$

(To see this, note that the slope at $t = t_0$ can be approximated by $g'(t_0) \approx \frac{g(t)-g(t_0)}{t-t_0}$.)

From $g'(t) = 2e^{2t}$, we have $g'(t_0) = 2e^{2t}|_{t=\frac{1}{2}\ln(2)} = 4$ and we know that, around $t = t_0$,

$$g(t) \approx g'(t_0)(t-t_0) + 0 = 4(t-t_0)$$

Therefore, the integral becomes

$$\int_0^1 \delta(e^{2t}-2)\,dt = \int_0^1 \delta\big(4(t-t_0)\big)\,dt\,.$$

Applying $\delta(at) = \frac{1}{|a|}\delta(at)$, we have

$$\int_0^1 \delta(e^{2t}-2)\,dt = \int_0^1 \frac{1}{|4|}\delta(t-t_0)\,dt = \frac{1}{4}\int_0^1 \delta(t-t_0)\,dt.$$

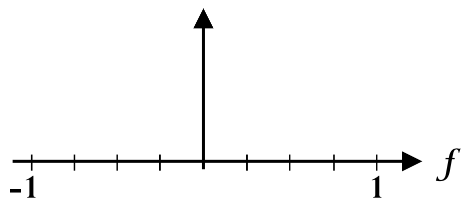
From the sifting property of the δ-function, because $t_0 \approx 0.3466$ is inside the integration range, the value of the integral is simply $\frac{1}{4}$.

2) [M2018] The impulse response of a multipath channel is of the form

$$h(t) = \sum_{k=1}^{\nu} \beta_k \delta(t - \tau_k).$$

Plot $|H(f)|$ when $\nu = 2,\ \beta_1 = \beta_2 = 0.5,\ \tau_1 = 1,\ \tau_2 = 3$

Consider the frequency from $f = -1$ to $f = 1$ Hz.



Here, we have $h(t) = 0.5\delta(t-1) + 0.5\delta(t-3)$.

We will present three solutions below. If you have familiarized yourselves with properties of Fourier transform, method 1 is easy and can quickly give you the answer. Method 3 tries to solve the problem directly without recalling any Fourier transform properties.

Method 1: Recall that

$$\cos(2\pi f_0 t) \xrightarrow{\mathcal{F}} \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0).$$

So, a cosine corresponds to two δ-functions of equal size in another domain. Here, we have two δ-functions in the time domain; so, we expect a cosine in the freq. domain. To reduce confusion, we will rename the constant f_0 to a:

$$\cos(2\pi a t) \xrightarrow{\mathcal{F}} \frac{1}{2}\delta(f - a) + \frac{1}{2}\delta(f + a).$$

From the above relation, applying duality theorem, we have

$$y(t) \equiv \frac{1}{2}\delta(t-a) + \frac{1}{2}\delta(t+a) \xrightarrow{\mathcal{F}} \cos\big(2\pi a(-f)\big) = \cos(2\pi a f) \equiv Y(f)$$

On the LHS, we have two δ-functions of equal size, centering around $t = 0$, with separation distance $= 2a$.

In this problem, $h(t) = y(t-2)$ and $a = \frac{2}{2} = 1$. By the time-shift property of Fourier transform,

$$H(f) = e^{-j2\pi(2)f}\, Y(f)$$

and

$$|H(f)| = |Y(f)| = |\cos(2\pi a f)| = |\cos(2\pi f)|.$$

Method 2: Recall, from our lecture on the two-path channel (Ex. 3.35):

When $h(t) = \beta_1\delta(t - \tau_1) + \beta_2\delta(t - \tau_2)$, we have

$$|H(f)|^2 = |\beta_1|^2 + |\beta_2|^2 + 2|\beta_1||\beta_2|\cos\big(2\pi(\tau_2 - \tau_1)f + (\phi_1 - \phi_2)\big),$$

where ϕ_1 and ϕ_2 are the phases of β_1 and β_2 respectively.

Here,

$$|H(f)|^2 = 0.5^2 + 0.5^2 + 2 \times 0.5^2 \cos(2\pi(2)f) = \frac{1}{2}(1 + \cos(2\pi(2)f)).$$

From $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$, we have

$$|H(f)| = \sqrt{\frac{1}{2}(1 + \cos(2\pi(2)f))} = \left|\cos\left(\frac{2\pi(2)f}{2}\right)\right| = |\cos(2\pi f)|.$$

Method 3: Using the time-shift property of Fourier transform, we have

$$\begin{aligned} H(f) &= 0.5e^{-j2\pi(1)f} + 0.5e^{-j2\pi(3)f} = e^{-j2\pi(2)f}\left(0.5e^{j2\pi(1)f} + 0.5e^{-j2\pi(1)f}\right) \\ &= e^{-j2\pi(1.5)f}\cos(2\pi f). \end{aligned}$$

Therefore,

$$|H(f)| = |\cos(2\pi f)|.$$

All methods give the same $|H(f)|$ below: